Topic 9 –
Variation of parameters

Topic 9 - Variation of Parameters

Topic 9 is another way to find y_p.

It will work in situations where topic 8 doesn't like

$$y'' + y = \tan(x)$$

Also topic 9 will let us solve equations where the coefficients aren't all constants like:

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

DERIVATION TIME

Suppose you already have the general solution to the homogeneous equation

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0}$$

and it is

$$y_h = c_1 y_1 + c_2 y_2$$

where y_1, y_2 are linearly independent.

Using y_1 and y_2 it is possible to find a solution to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = b(x)}$$

To do this set

$$y_p = v_1 \boxed{y_1} + v_2 \boxed{y_2}$$

these are homogeneous solutions

Where v_1, v_2 are unknown functions to be determined.

We will plug this in and make it work!

We need the derivatives:

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

$$= (v_1 y_1' + v_2 y_2') + \underbrace{(v_1' y_1 + v_2' y_2)}_{\text{assume } 0}$$

To simplify assume

$$\boxed{v_1' y_1 + v_2' y_2 = 0}$$

So we have

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1 y_1' + v_2 y_2'$$

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Plug these into

$$y'' + a_1(x) y' + a_0(x) y = b(x)$$

to get

$$\left(v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''\right) \leftarrow y_p''$$

$$+ a_1(x)\left(v_1 y_1' + v_2 y_2'\right) \leftarrow + a_1(x) y_p'$$

$$+ a_0(x)\left(v_1 y_1 + v_2 y_2\right) \leftarrow + a_0(x) y_p$$

$$= b(x) \leftarrow = b(x)$$

This becomes:

$$v_1\big(\underbrace{y_1'' + a_1(x) y_1' + a_0(x) y_1}_{0}\big)$$

$$+ v_2\big(\underbrace{y_2'' + a_1(x) y_2' + a_0(x) y_2}_{0}\big)$$

$$+ \left(v_1' y_1' + v_2' y_2'\right) = b(x)$$

The above are 0 because y_1, y_2 solve $y'' + a_1(x) y' + a_0(x) y = 0$

We are left with

$$\boxed{v_1' y_1' + v_2' y_2' = b(x)}$$

Summarizing we must solve the following for v_1', v_2':

$$\boxed{\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \qquad (1) \\ v_1' y_1' + v_2' y_2' &= b(x) \qquad (2) \end{aligned}}$$

(1) simplifying assumption

(2) got from derivation

To solve for v_2' we can calculate $y_1' * (1) - y_1 * (2)$ to get:

$$(\cancel{y_1' v_1' y_1} + y_1' v_2' y_2) - (\cancel{y_1 v_1' y_1'} + y_1 v_2' y_2')$$

$$= y_1' \cdot 0 - y_1 \cdot b(x)$$

We get

$$y_1' v_2' y_2 - y_1 v_2' y_2' = -y_1 b(x)$$

So,

$$v_2' = \frac{-y_1 b(x)}{y_1' y_2 - y_1 y_2'}$$

Then

$$v_2' = \frac{y_1 b(x)}{-(y_1' y_2 - y_1 y_2')}$$

$$= \frac{y_1 b(x)}{y_1 y_2' - y_1' y_2}$$

$$= \frac{y_1 b(x)}{W(y_1, y_2)}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Then,
$$v_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)}\, dx$$

To find v_1 we calculate $y_2' * ① - y_2 * ②$ to get:

$$\underbrace{y_2' v_1' y_1 + \cancel{y_2' v_2' y_2}}_{y_2' * ①} \underbrace{- y_2 v_1' y_1' - \cancel{y_2 v_2' y_2'}}_{-y_2 * ②} = 0 - y_2 b(x)$$

giving $v_1'(y_2' y_1 - y_2 y_1') = -y_2 b(x)$

so $v_1' = \dfrac{-y_2 b(x)}{y_2' y_1 - y_2 y_1'}$

or $v_1 = \displaystyle\int \frac{-y_2 b(x)}{W(y_1, y_2)}$

Below is a summary.

Summary

Suppose you have two linearly independent solutions y_1, y_2 to

$$\boxed{y'' + a_1(x) y' + a_0(x) y = 0}$$

Then a particular solution to

$$\boxed{y'' + a_1(x) y' + a_0(x) y = b(x)}$$

is given by

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} dx$$

Ex: Solve

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Step 1: Solve

$$y'' - 4y' + 4y = 0$$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

$$r = 2$$

repeated root

So,

$$\boxed{y_h = c_1 e^{2x} + c_2 x e^{2x}}$$

We will use $y_1 = e^{2x}$, $y_2 = xe^{2x}$ in step 2.

Step 2: Now we find y_p for

$$y'' - 4y' + 4y = \underbrace{(x+1)e^{2x}}_{b(x)}$$

We need the Wronskian.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + x(2e^{2x}) \end{vmatrix}$$

$$= (e^{2x})(e^{2x} + 2xe^{2x})$$
$$- (xe^{2x})(2e^{2x})$$

$$= e^{4x} + \cancel{2x e^{4x}}$$
$$- \cancel{2x e^{4x}}$$

$$= e^{4x}$$

We have

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{-(xe^{2x})(x+1)e^{2x}}{e^{4x}} dx$$

$$= \int \frac{-x(x+1)e^{2x}e^{2x}}{e^{4x}} dx$$

$$= \int \frac{-x(x+1)\cancel{e^{4x}}}{\cancel{e^{4x}}} dx$$

$$= \int (-x^2 - x)\, dx$$

$$= \boxed{-\frac{1}{3}x^3 - \frac{1}{2}x^2} \leftarrow v_1$$

We also have

$$v_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)}\, dx$$

$e^{2x} e^{2x} = e^{4x}$

$$= \int \frac{\cancel{e^{2x}}(x+1)\cancel{e^{2x}}}{\cancel{e^{4x}}}\, dx$$

$$= \int (x+1)\, dx$$

$$= \boxed{\frac{1}{2}x^2 + x} \leftarrow v_2$$

So,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Step 3: The general solution to

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

is

$$y = y_h + y_p$$

$$= c_1 e^{2x} + c_2 x e^{2x}$$

$$+ \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Ex: Solve

$$y'' + y = \tan(x)$$

Step 1: Solve

$$y'' + y = 0$$

The characteristic equation is

$$r^2 + 1 = 0$$

The roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{\pm\sqrt{-4}}{2} = \frac{\pm \cancel{\sqrt{4}}\sqrt{-1}}{\cancel{2}}$$

$$= \pm\sqrt{-1} = \pm i$$

$$= \underbrace{0 \pm 1 \cdot i}_{\alpha \pm \beta i}$$

So,

$$y_h = \underbrace{c_1 e^{0x} \cos(1 \cdot x) + c_2 e^{0x} \sin(1 \cdot x)}_{c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)}$$

$$\boxed{y_h = c_1 \cos(x) + c_2 \sin(x)}$$

$e^{0x} = e^0 = 1$

$y_1 = \cos(x)$

$y_2 = \sin(x)$

For step 2

Step 2: Find y_p for

$$y'' + y = \tan(x)$$

We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \left(\cos(x)\cos(x)\right) - \left(-\sin(x)\right)\left(\sin(x)\right)$$

$$= \cos^2(x) + \sin^2(x)$$

$$= 1$$

We have

$$v_1 = \int \frac{-y_2 \, b(x)}{W(y_1, y_2)} \, dx$$

$$= \int \frac{-\sin(x)\tan(x)}{1}\,dx$$

$$= \int -\sin(x) \cdot \frac{\sin(x)}{\cos(x)}\,dx$$

$$= \int \frac{-\sin^2(x)}{\cos(x)}\,dx$$

$$= \int \frac{\cos^2(x)-1}{\cos(x)}\,dx$$

$$\cos^2(x)+\sin^2(x)=1$$
$$-\sin^2(x)=\cos^2(x)-1$$

$$= \int \left(\frac{\cos^2(x)}{\cos(x)} - \frac{1}{\cos(x)}\right)dx$$

$$= \int \left(\cos(x) - \sec(x) \right) dx$$

$$= \underbrace{\boxed{\sin(x) - \ln|\sec(x) + \tan(x)|}}_{v_1}$$

And

$$v_2 = \int \frac{y_1 \, b(x)}{W(y_1, y_2)} \, dx$$

$$= \int \frac{\cos(x) \tan(x)}{1} \, dx$$

$$= \int \cancel{\cos(x)} \cdot \frac{\sin(x)}{\cancel{\cos(x)}} \, dx$$

$$= \int \sin(x) \, dx = \underbrace{\boxed{-\cos(x)}}_{v_2}$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(\sin(x) - \ln|\sec(x) + \tan(x)|\right)\cos(x)$$

$$+ \left(-\cos(x)\right)\sin(x)$$

$$= \underbrace{\boxed{-\ln|\sec(x) + \tan(x)| \cdot \cos(x)}}_{y_p}$$

Step 3: The general solution to

$$y'' + y = \tan(x)$$

is

$$y = y_h + y_p$$

$$= c_1 \cos(x) + c_2 \sin(x)$$
$$- \ln|\sec(x) + \tan(x)| \cdot \cos(x)$$